

# Optimal Quantum Estimation for Gravitation

T. G. Downes and G. J. Milburn

Centre for Engineered Quantum Systems,  
School of Mathematics and Physics,  
The University of Queensland, St Lucia, Australia

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## Abstract

Here we describe the quantum limit to estimation of the spacetime metric, or equivalently the quantum limit to measuring the classical gravitational field. Specifically, we write down the optimal quantum Cramer-Rao lower bound, for any single parameter describing a metric for spacetime. Four key examples are demonstrated covering a broad range of relativistic phenomena. We describe quantum limited estimation of the mass of a black-hole, the acceleration of a uniformly accelerating observer, the amplitude of a gravitational wave and the expansion parameter in a cosmological model. The standard time-energy uncertainty relation and the Heisenberg uncertainty relation are special cases of the uncertainty relation for the spacetime metric. The uncertainty relation takes a particularly simple and revealing form when the measurement region is made sufficiently small. We use the locally covariant formulation of quantum field theory in curved spacetime, which allows for a manifestly spacetime independent derivation. The result is an uncertainty relation applicable to all causal spacetime manifolds.

## 1 Introduction

The geometry of spacetime is determined by physical measurements made with clocks and rulers or, more generally, using quantum fields, sources and

detectors. As these are physical systems, the ultimate accuracy achievable is determined by quantum mechanics. In this paper we obtain a parameter-based quantum uncertainty relation, bounding knowledge of the spacetime metric.

The gravitational field is treated entirely in accordance with classical general relativity. In general relativity, the gravitational field is a manifestation of the geometry of spacetime, which in turn is described by a metric. The metric essentially gives the proper times and proper distances between events in spacetime. The infinitesimal proper distance between two spacetime events is given by [1]:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad (1)$$

Where  $g_{\mu\nu}(x)$  is the metric tensor, with indices  $\mu, \nu = 0, 1, 2, 3$  for the time and three spatial components. The  $dx^\mu$  are the infinitesimal coordinate distances. We assume the Einstein summation convention, where repeated upper and lower indices, are to be summed over, for example in equation (1).

We know from Heisenberg that there is an uncertainty relation between position and momentum. It states that the product of the uncertainty in position and the uncertainty in momentum, must always be greater than a constant. We write down Heisenberg's uncertainty relation, in terms of the variances of position and momentum:

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \frac{\hbar^2}{4}$$

This relationship bounds knowledge of the position and momentum of a quantum system, where  $\hbar$  is given by the reduced Planck's constant. The Heisenberg uncertainty relation is derived in standard quantum mechanics, where position and momentum are both represented as Hermitian operators.

A similar relation also exists between time and energy:

$$\langle(\delta t)^2\rangle\langle(\Delta\hat{H})^2\rangle \geq \frac{\hbar^2}{4}$$

Unlike position, time in standard quantum mechanics is given by a classical parameter  $t$  with classical variance  $\langle(\delta t)^2\rangle$ . The time-energy uncertainty relation can be considered as an example of quantum parameter estimation. There, one tries to estimate a classical parameter, in this case time, by making

measurements on a quantum system, which depends on the parameter of interest.

The most common way to make quantum mechanics compatible with classical relativity, is to demote position to a parameter, just like time is in the previous example. Physical systems can then be thought of as living on, and interacting with, the classical spacetime manifold. In this relativistic context, the Heisenberg uncertainty relation should also be framed in the parameter estimation context.

This way of thinking was used by Braunstein, Caves and Milburn [2], to develop optimal quantum estimation for spacetime displacements, in flat Minkowski spacetime. In spacetime, not only can one move a fixed proper distance or time, one can also boost and rotate. Quantum parameter estimation was thus also developed for the parameters corresponding to these actions [2]. The results were developed with the quantised electromagnetic field as the measurement system. They show that the estimate of spacetime translations may be made more accurate, if the uncertainty in the number operator is made very large.

Before we describe quantum parameter estimation in more detail, we ask the question: can any insight be found by simply applying the Heisenberg uncertainty relation, in its parameter based form, directly to a proper distance? It was in this way that Unruh [3] derived an uncertainty principle for the  $g_{11}$  component of the metric tensor. The calculation was simple and made the important point that in general relativity the coordinate system is arbitrary and so there should only be quantum uncertainty in the proper time and the proper distance. As these are in turn related to the metric via equation (1), any uncertainty in the proper distance is equivalent to uncertainty in the metric. By applying the Heisenberg uncertainty relation to a proper distance, a simple but insightful uncertainty relation was found for one component of the metric. We write down the Unruh uncertainty relation for the  $xx$  component of the metric, in terms of variances as:

$$\langle(\delta g_{11})^2\rangle\langle(\Delta\hat{T}^{11})^2\rangle\geq\frac{\hbar^2}{V^2}$$

The key finding of this uncertainty relation is the inverse proportionality to  $V^2$ , the square of the four-volume of the measurement. The conjugate variable to  $g_{11}$  is the corresponding component of the quantised stress-energy tensor  $\hat{T}^{11}$ , in this case the pressure in the  $x$ -direction.

In this paper we consider a more general context for deriving an uncertainty relation for the metric, by formulating it as a problem in quantum estimation theory. The metric  $g_{\mu\nu}(x)$  is defined for each point  $x$  on the manifold. If our quantum measurement system occupies some four-volume  $V$ , then the system will depend on the metric at every point in that region. If we consider the metric to be an arbitrary function on the manifold, then we need to estimate an infinite number of parameters to define it completely. Instead we will consider regions of spacetime, which can be described by metrics, defined by a finite number of parameters  $\theta_1, \theta_2 \dots \theta_N$ . For example the Schwarzschild metric, which describes the spacetime around a static non-rotating blackhole, is defined by only one parameter  $M$ , the mass of the blackhole. The task is to estimate the parameters by making measurements on physical systems living on the spacetime manifold.

The general schema for quantum parameter estimation is represented in figure 1. An initial quantum state, represented by a density operator  $\rho_0$ , is

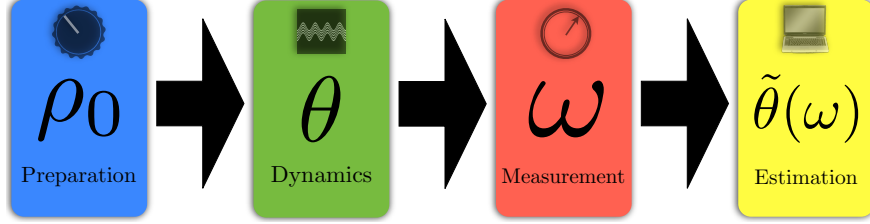


Figure 1: The scheme for quantum parameter estimation.

evolved by a unitary transformation  $\hat{U}(\theta)$  dependant on the parameter  $\theta$  of interest. Measurements are then made on the system, producing measurement results  $\omega$ , which are in turn fed into an estimator  $\tilde{\theta}(\omega)$  of the parameter.

We consider generalised measurements, known as positive operator valued measures. These are Hermitian operators with the property:

$$\int \hat{E}(\omega) d\omega = \mathbb{1}$$

Where  $\omega$  is the measurement result and  $\mathbb{1}$  is the identity operator. The outcomes of a particular measurement will follow a probability distribution  $p(\omega|\Theta)$  conditional on the actual value of the parameter  $\Theta$ . The probability distribution for the outcomes  $\omega$  can be calculated as:

$$p(\omega|\Theta) = \text{Tr} \left( \hat{\rho}(\Theta) \hat{E}(\omega) \right) \quad (2)$$

Where  $\hat{\rho}(\Theta) = \hat{U}(\Theta)\rho_0\hat{U}^\dagger(\Theta)$  is the state after the  $\theta$  dependant interaction.

The problem of estimating the parameter  $\theta$  is essentially that of choosing a value  $\tilde{\theta}$  to fit the probability distribution  $p(\omega|\tilde{\theta})$  to the actual measured values. A common example is the so-called, maximum likelihood estimator, which is the choice of  $\tilde{\theta}$  which retrospectively maximises the probability, of the observed measurement values.

The variance of any estimate of the parameter  $\theta$ , based on the distribution of the observed measurements, will be bounded below by the classical Cramer-Rao lower-bound [2]:

$$\langle(\delta\tilde{\theta})^2\rangle \geq \frac{1}{F(\Theta)}$$

where  $F(\Theta)$  is the Fisher information for the measurement, given by:

$$F(\Theta) = \int d\omega \frac{1}{p(\omega|\Theta)} \left( \frac{\partial p(\omega|\theta)}{\partial \theta} \Big|_{\theta=\Theta} \right)^2$$

The key ingredient is the rate of change of the probability distribution with respect to the parameter. A larger response due to a change in the parameter naturally gives rise to better estimation of the parameter. The rate of change of the probability distribution can be calculated as:

$$\frac{\partial p(\omega|\theta)}{\partial \theta} = \text{Tr} \left( i[\hat{\rho}(\theta), \hat{h}] \hat{E}(\omega) \right)$$

Where  $\hat{h}$  is the generator of the unitary transformation  $U(\theta)$ . We have described quantum parameter estimation in the Schrödinger picture where the density operator depends on the parameter. Due to cyclicity of the trace used to calculate both the probability distribution and its rate of change, the Heisenberg picture can equally be used, which can be quite useful in the relativistic setting. The key component is the generator  $\hat{h}$  which generates changes in the probability distributions due to changes in the parameter. This can be seen directly for pure states, when one optimises over all possible measurements. The optimal quantum Fisher information is then given by:

$$F(\Theta) = \frac{4\langle(\Delta\hat{h}[\Theta])^2\rangle}{\hbar^2}$$

Where  $\langle(\Delta\hat{h}[\Theta])^2\rangle$  is the quantum variance of the generator  $\hat{h}$  taken in the initial state. The optimal quantum Cramer-Rao lower bound can then be expressed as:

$$\langle(\delta\tilde{\theta})^2\rangle\langle(\Delta\hat{h}[\Theta])^2\rangle \geq \frac{\hbar^2}{4}$$

One can now easily construct examples by simply identifying parameters and their corresponding generators. For example, the Hamiltonian is the generator of time translations, which gives the uncertainty relation presented in the introduction. Another important example is the number operator and phase which is the basis of Heisenberg limited phase estimation.

## 2 An Uncertainty Relation for the metric $g_{\mu\nu}$

The physical systems we consider here are quantum fields, for example the Dirac field for electrons. For clarity we restrict to the free scalar field. The generalisation to interacting and higher spin fields, should follow a similar argument to the one we present here for the scalar field. We consider only measurements on scales large enough, such that the quantisation of the gravitational field itself can be safely ignored. We also work in the test-field approximation, where the gravitational field of the probe is ignored completely. For example one might choose to ignore the gravitational field of the laser in a gravitational wave interferometer. It is important to note however, that the action of the probe fields on the spacetime manifold will only strengthen the bound we derive below. That is, if a high energy field were used to probe the structure of spacetime, then the back-reaction of the field on the spacetime, would further limit the accuracy of the measurements, preventing our inequalities from being saturated. The problem of back-reaction when measuring the structure of spacetime has already been studied in some detail [4].

Apart from the test-field approximation, the quantum theory for the fields we consider here takes classical general relativity fully into account. We employ the locally covariant formulation of quantum field theory on curved spacetime [5]. Local covariance refers to the global property, of having the same physics under all coordinate systems, i.e. general covariance, together with the local property that if two localised spacetime regions are equivalent, then they should describe the same physics regardless of whats happening at

distant regions of the universe. In this sense the locally covariant formulation fully takes into account the locality and covariance of general relativity. We only omit causality-violating spacetimes, which removes certain causal pathologies associated with closed time-like curves and time machines. Our results will hold for all causal spacetime manifolds.

A key result of the locally covariant approach, is the calculation of how quantum observables respond to changes in the metric. Say we believe some particular region of the universe to be well described by a metric  $g_{\mu\nu}(\theta)$  depending on  $N$  parameters  $\theta = (\theta_1, \theta_2, \dots, \theta_N)$ . For this case the locally covariant approach can be used to calculate how any observable  $\hat{E}(\theta)$  will respond to a change in any one of the parameters. The response is evaluated as the rate of change of the observable with respect to the parameter and as we noted in the introduction, this is just what we need to calculate the quantum Cramer-Rao lower bound.

We first consider an arbitrary region of spacetime and divide it into three sub-regions as shown in figure 1. The lower region labeled  $\mathcal{M}^-$  is the region where the field is prepared. It is in the causal past of the upper region labeled  $\mathcal{M}^+$  where measurements are performed. The intermediate region  $\mathcal{M}$  is inside the intersection of the causal past of  $\mathcal{M}^+$  and the causal future of  $\mathcal{M}^-$ .

We consider a diffeomorphism  $\phi_s$  which smoothly deforms the metric inside the region  $\mathcal{M}$  and acts like the identity everywhere else. The effect of this change on any observable  $\hat{E}(\theta)$  in the measurement region can be calculated by the action of a map, known as the relative Cauchy evolution [5]. First we construct the observable  $\hat{E}(\theta)$  for a particular value of the parameter,  $\theta = \Theta$ , and then use the relative Cauchy evolution to calculate the rate of change with respect to  $\theta$ . Specifically the diffeomorphism chosen acts on the metric in the region of interest as:

$$\phi_s g_{\mu\nu}(\Theta) = g_{\mu\nu}(\Theta + s)$$

The number  $\Theta$  can be thought of as the actual or expected value of the parameter. If we have an unbiased estimator  $\tilde{\theta}$  of the parameter  $\theta$  then the expected value of the estimate  $\langle \tilde{\theta} \rangle = \Theta$ .

We construct the measurement operators out of polynomials of localised field operators. For a particular measurement  $\hat{E}(\omega|\theta)$ , conditional on a parameter  $\theta$  in the metric, we can construct the probability distribution of the

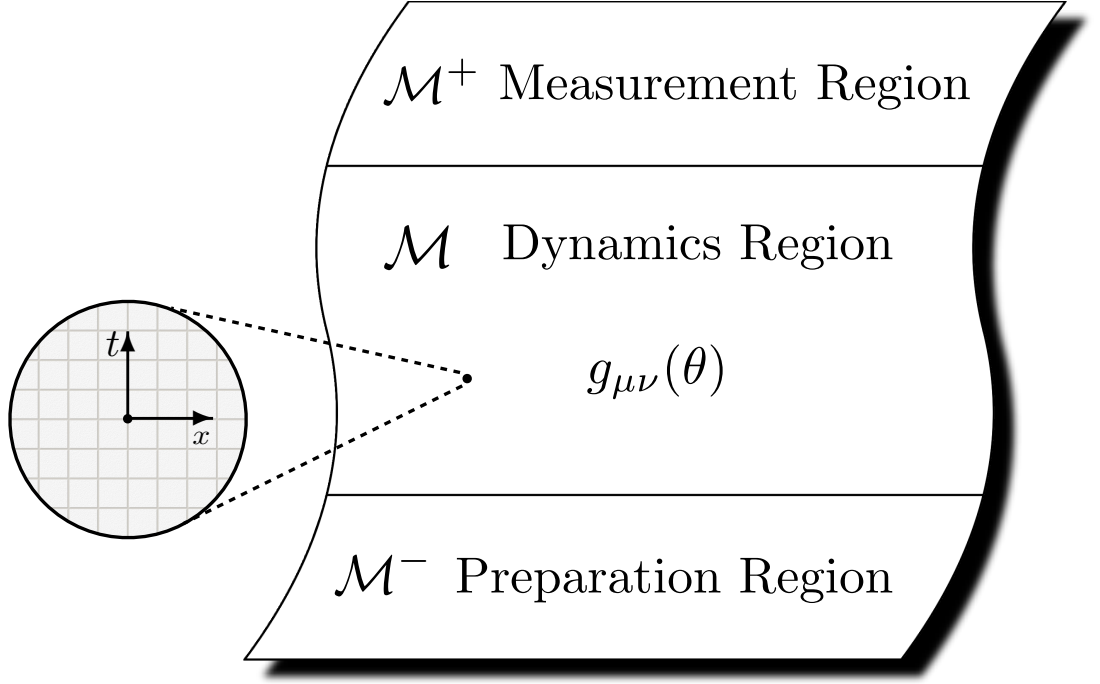


Figure 2: Spacetime diagram. The middle region is described by a metric  $g_{\mu\nu}(\theta)$  and it is here that one wishes to obtain the value of  $\theta$ .

outcomes  $\omega$  as:

$$p(\omega|\theta) = \text{Tr} \left( \hat{\rho} \hat{E}(\omega|\theta) \right)$$

where  $\hat{\rho}$  is the density operator representing the state of the system. The relative Cauchy evolution gives the rate of change of the probability distribution with respect to the parameter as [5]:

$$\left. \frac{dp(\omega|\theta)}{d\theta} \right|_{\theta=\Theta} = \text{Tr} \left( i\hat{\rho} [\hat{E}(\omega|\Theta), \hat{P}(\Theta)] \right) \quad (3)$$

where the operator  $\hat{P}(\theta)$  is given by:

$$\hat{P}(\theta) = \frac{1}{2} \int_{\mathcal{M}} d\mu \hat{T}^{\mu\nu} \frac{dg_{\mu\nu}(\theta)}{d\theta}$$



Here  $\hat{T}^{\mu\nu}$  is the renormalised stress-energy tensor of the scalar field used by the measuring device.  $\mathcal{M}$  is the region of interest and  $d\mu$  is the volume-form induced by  $g_{\mu\nu}(\Theta)$ . We identify the operator  $\hat{P}(\theta)$  as the infinitesimal generator of changes in the probability distribution due to changes in the parameter  $\theta$ . The cyclicity of the trace and the simple commutator form of the expression in relation (1) means that the dependance can either be thought of as in the observable or the state. The standard techniques of quantum parameter estimation, as described in the introduction, can be immediately applied. For pure states, optimisation over all possible measurements, yields the following uncertainty relation:

$$\langle(\delta\tilde{\theta})^2\rangle\langle(\Delta\hat{P}[\Theta])^2\rangle\geq\frac{\hbar^2}{4}\quad(4)$$

This relation gives the optimal quantum limit to measuring any parameter of a spacetime metric. It is locally and covariantly defined and applies to all causal spacetime manifolds. To demonstrate the ease with which it can be applied, we shall now present a number of examples.

### 3 Estimating the Mass of a Black Hole

As our first example, we shall consider the problem of measuring the gravitational field, outside a spherically symmetric, non-rotating, massive body. The metric for the empty space outside such an object, is given by the Schwarzschild solution to Einstein's field equations [1]. In Schwarzschild coordinates, with units such that the speed of light  $c = 1$ , the metric becomes:

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (5)$$

where  $M$  is the mass as observed from infinity and  $G$  is the gravitational constant. These coordinates have the intuitive features that; surfaces of constant  $t$  and  $r$  have area given by  $4\pi r^2$  and the metric (5) becomes the metric of an inertial observer in flat spacetime, as  $r$  becomes large. We choose the region of interest  $\mathcal{M}$  to have duration  $L_t = t_2 - t_1$ , length  $L_r = r_2 - r_1$  and solid angle  $L_\Omega = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \sin \theta d\theta d\phi$  in Schwarzschild coordinates.

The uncertainty relation (4) is defined in terms of a four-volume integral of a scalar quantity. Due to this, it is invariant under general coordinate

transformations. We shall take full advantage of this invariance and change to more well behaved coordinates, in order to evaluate the uncertainty relation. We transform to outgoing Eddington-Finkelstein coordinates, which are appropriate for describing photons propagating out of the gravitational potential [1]. The outgoing null geodesics are labeled with a new coordinate  $\tilde{U} = t - r^*$  where:

$$r^* = r + 2MG \ln |r/2MG - 1|$$

In the coordinates  $(\tilde{U}, r, \theta, \phi)$  the metric takes the outgoing Eddington-Finkelstein form [1]:

$$ds^2 = - \left( 1 - \frac{2MG}{r} \right) d\tilde{U}^2 - 2d\tilde{U}dr + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

For this example we shall choose a state, such that the variance of the stress-energy tensor, is constant throughout the measurement region, when expressed in Eddington-Finkelstein coordinates. In a different state the variance might not be constant in these coordinates. The behaviour of the variance, over the measurement region, will affect the explicit dependence of the uncertainty relation, on the shape and size of the measurement region. With the choice of state and measurement region we have made, the uncertainty relation (4) becomes:

$$\langle (\delta \tilde{M})^2 \rangle \langle (\Delta \hat{T}^{00})^2 \rangle \geq \left( \frac{\hbar/c}{GL_t L_\Omega [r_2^2 - r_1^2]} \right)^2$$

where we have reinserted the correct factors of  $c$ . We see that the Fisher information depends on the variance of the energy-density, in Eddington-Finkelstein coordinates. The Fisher information also increases, as one increases the size of the measurement region. This makes for an increased effect of the mass  $M$  and hence a better estimate. The dependance on the distance from the mass is clearly stronger, than that of time or solid angle. This is due to the fact that the gravitation field varies in this direction. One achieves better estimation therefore, by extending the apparatus out in this direction. By defining our measurement region in Schwarzschild coordinates, we have chosen a particular shape for the apparatus. Although the Fisher information does not depend on the coordinates, it will depend on the shape of the region. As we shall see, the underlying dependance on the four-volume of the measurement, will be the inverse dependence seen in the Unruh relation (3).

## 4 Estimation of Proper Acceleration

We now give an example, for the use of the uncertainty relation (4) in flat spacetime. In this example it is the property of an observer we are interested in, not the spacetime itself and so the observer effectively fixes the coordinate system. This example is perhaps best interpreted as a constant acceleration drive spacecraft, trying to determine it's own acceleration by local measurements. The natural coordinate system, for a uniformly accelerating observer, is known as the Fermi-Walker transported orthonormal tetrad. It is the coordinate system carried by the accelerating observer, in which he is instantaneously at rest. This coordinate system has limits, it can only be extended at most a distance  $a^{-1}$  from the observer, where  $a$  is the proper acceleration. The flat Minkowski metric in these coordinates can be written as [1]:

$$ds^2 = -(1 + a\xi^1)^2(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2$$

Here we take the region of interest to be a 4-cube, centred at the origin, in these coordinates with sides of length  $L_{\xi^1}, L_{\xi^2}, L_{\xi^3}$  and duration  $L_{\xi^0}$ . Once again we choose a state where the variance of the stress-energy tensor is constant, in the observers coordinates. The uncertainty relation for the proper acceleration  $a$  is then given by:

$$\langle(\delta\tilde{a})^2\rangle\langle(\Delta\hat{T}^{00})^2\rangle \geq \left(\frac{3\hbar}{aL_{\xi^0}L_{\xi^1}^3L_{\xi^2}L_{\xi^3}}\right)^2 \quad (6)$$

We see that better estimation is achieved by making the apparatus larger and having a bigger acceleration. However, if we make the apparatus as large as possible, by inserting the constraint  $\xi_1 < \frac{1}{a}$  we find:

$$\langle(\delta\tilde{a})^2\rangle\langle(\Delta\hat{T}^{00})^2\rangle \geq \left(\frac{3\hbar a^2}{L_{\xi^0}L_{\xi^2}L_{\xi^3}}\right)^2 \quad (7)$$

Now the inverse is true, if one has reached an acceleration, such that the size of the apparatus has been limited, then larger accelerations will produce worse estimation. This can be seen in the first instance, from the stronger dependence on size as compared to acceleration.

## 5 Quantum Limited Gravitational Wave Detection

We now consider another example of the uncertainty relation (4), in estimating the amplitude of a gravitational wave. In the linear approximation of Einstein's equations, the metric can be written as the flat Minkowski metric plus a small perturbation.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The simplest solutions to the linearised Einstein equations are plane-waves. We write down these wave solutions, in the transverse traceless gauge, as [1]:

$$h_{\mu\nu} = \mathbb{R}[A_{\mu\nu} \exp(k_a x^a)] \quad (8)$$

The amplitude of the wave  $A_{\mu\nu}$  has only two independent components corresponding to the two polarisation states. For our example here, we shall consider a gravitational wave propagating in the  $z$  direction and linearly polarised along the diagonal directions in the  $x$ - $y$  plane. For this case the only non-zero components of the metric (8) are [1]:

$$h_{xy} = h_{yx} = \mathbb{R}[A_{\times} e^{-i\omega(t-z)}]$$

As the wave is a small perturbation  $A_{\times} \ll 1$  we approximate the four-volume element, as being the volume element of the flat metric. For our measurement region we take a 4-cube centred at the origin, in Minkowski coordinates, with length  $L_x, L_y, L_z$  and duration  $L_t$ . Again we choose a state where the variance of the stress-energy tensor is constant. When estimating the amplitude the uncertainty relation then becomes:

$$\langle (\delta A_{\times})^2 \rangle \langle (\Delta \hat{T}^{xy})^2 \rangle \geq \frac{\hbar^2 \omega^4}{64 L_x^2 L_y^2 \sin^2[\frac{1}{2}\omega L_t] \sin^2[\frac{1}{2}\omega L_z]}$$

One can see that larger frequency waves will produce worse estimates for the amplitude. Also the Fisher information oscillates to zero whenever the transverse length or duration of the apparatus is a multiple of the wavelength. Consider a ring of atoms sitting in the plane perpendicular to the direction of propagation of the wave. Once a complete wavelength has passed they will distort and then return to their original position. If a measurement is only

made at this point then no effect will be observed, and hence no information on the amplitude will have been received. A limitation to the uncertainty relation (4) in dynamical situations like this one, is that it is based on a single measurement. Generalising it to take into account continuous measurements is then the logical next step. It will require analysing the Fisher information matrix and will result in a spectral uncertainty relation [6].

## 6 Estimating the expansion of the universe

For our final example we shall consider the spatially flat Friedmann-Robertson-Walker cosmology. This is a universe filled with a uniform density of galaxies. At any instant in time, in the co-moving frame of the galaxies, the universe looks the same everywhere (homogeneous) and in all directions (isotropic). The metric for this universe is given by [1];

$$ds^2 = -dt^2 + a^2(t) [dx^2 + dy^2 + dz^2]$$

where  $t$  is the proper time of an observer co-moving with any of the galaxies. The spatial coordinates  $x, y, z$  describe the homogeneous and isotropic surfaces of constant proper time  $t$ . The function  $a(t)$ , known as the expansion parameter, is the ratio of the proper distance between any two galaxies at the initial time  $t = 0$  and the time  $t$ .

During an infinitesimal duration of proper time  $dt$  a photon will travel the distance  $d\eta = \frac{dt}{a(t)}$ . It is convenient to use this as the time parameter. We shall consider a universe dominated by matter, in which case the metric becomes;

$$ds^2 = \frac{a_{\max}^2}{4} (1 - \cos \eta)^2 [-d\eta^2 + dx^2 + dy^2 + dz^2]$$

where  $\eta$  runs between 0 at the beginning of expansion to  $2\pi$  at the end of recontraction. We wish to estimate the parameter  $a_{\max}$  which controls the maximum size the universe reaches before recontraction commences. The uncertainty relation for this parameter becomes:

$$\sum_{\mu} \langle (\delta \tilde{a}_{\max})^2 \rangle \langle (\Delta \hat{T}^{\mu\mu})^2 \rangle \geq \left( \frac{16\hbar}{L_x L_y L_z a_{\max}^5 \int_{\eta_1}^{\eta_2} d\eta (1 - \cos \eta)^6} \right)^2$$

The integral can be performed analytically but perhaps becomes somewhat clearer if expressed in terms of the density of mass-energy  $\rho$ :

$$\sum_{\mu} \langle (\delta \tilde{a}_{max})^2 \rangle \langle (\Delta \hat{T}^{\mu\mu})^2 \rangle \geq \left( \frac{4\hbar}{3\pi L_x L_y L_z a_{max} \int_{\eta_1}^{\eta_2} d\eta \rho^{-2}(\eta)} \right)^2$$

Better estimation is made by running the experiment for longer time  $\eta$ . However for fixed duration, better estimation is made during the period of low energy-density, which occurs at the point of maximum expansion.

## 7 Estimation of Proper Time and Proper Distance

Consider now making measurements in a local inertial frame, sufficiently small such that the metric can be approximated as constant, over the measurement region. The only parameters to estimate are then the components of the metric. For the first component of the metric, the uncertainty relation (9) becomes:

$$\langle (\delta g_{00})^2 \rangle \langle (\Delta \int_{\mathcal{M}} d\mu \hat{T}^{00})^2 \rangle \geq \hbar^2 \quad (9)$$

The integral of the energy density over the spatial component of the four volume is the Hamiltonian:

$$\int_{\mathcal{M}} d\mu \hat{T}^{00} = \sqrt{\langle g_{00} \rangle} \int dt \hat{H}(t)$$

and so the uncertainty relation (9) becomes:

$$\langle (\delta g_{00})^2 \rangle \langle (\Delta \int dt \hat{H}(t))^2 \rangle \geq \frac{\hbar^2}{\langle g_{00} \rangle} \quad (10)$$

Now consider the proper time of an observer stationary in this inertial coordinate system. The proper time in this case is given by:

$$\tau = t \sqrt{g_{00}}$$

The coordinates have been chosen so there is no uncertainty in the coordinate time. Using the delta method, the uncertainty in the metric is then related to the uncertainty in the proper time by:

$$\langle(\delta g_{00})^2\rangle = 4\langle g_{00}\rangle\langle(\delta\tau)^2\rangle/t^2$$

Inserting this relation into equation (10) gives the time energy uncertainty relation generalised to curved spacetime in a local Lorentz frame:

$$\langle(\delta\tau)^2\rangle\langle(\Delta\frac{1}{t}\int dt\hat{H}(t))^2\rangle \geq \frac{\hbar^2}{4\langle g_{00}\rangle^2} \quad (11)$$

If we assume a time independent Hamiltonian and specialise to flat spacetime the uncertainty relation (11) reduces to:

$$\langle(\delta\tau)^2\rangle\langle(\Delta\hat{H})^2\rangle \geq \frac{\hbar^2}{4} \quad (12)$$

This demonstrates that the standard time energy uncertainty relation is a special case of the metric uncertainty relation (4). Using a similar argument one can also derive an uncertainty relation for proper distance  $X$ .

$$\langle(\delta X)^2\rangle\langle(\Delta\hat{P})^2\rangle \geq \frac{\hbar^2}{4} \quad (13)$$

This is the parametric version of the Heisenberg uncertainty relation where  $\hat{P}$  is the momentum in the direction of the displacement.

## 8 Simplified Metric Uncertainty Relation

Let us consider the uncertainty relation (9) for an arbitrary component of the metric:

$$\langle(\delta g_{\mu\nu})^2\rangle\langle(\Delta\int_{\mathcal{M}} d\mu\hat{T}^{\mu\nu})^2\rangle \geq \hbar^2 \quad (14)$$

Where in this case there is no sum over the repeated indices. We further simplify the uncertainty relation (14) by assuming the stress-energy tensor is also constant over the measurement region. In this case one is left with a particularly simple form for the metric uncertainty relation given by:

$$\langle(\delta g_{\mu\nu})^2\rangle\langle(\hat{T}^{\mu\nu})^2\rangle \geq \frac{\hbar^2}{V^2} \quad (15)$$

where  $V$  is the four-volume of the measurement. This form of the fundamental dependance confirms the earlier relation (3) found by Unruh. Indeed the same restrictions were needed to derive (3) as were used to produce (15).

## 9 Conclusion

In this paper we have presented the optimal quantum Cramer-Rao lower bound for parameters describing a metric for spacetime. Our specific derivation applies for pure states of the scalar field on an arbitrary causal spacetime manifold. We give four important examples covering the full gamut of relativistic phenomena. We described quantum estimation for; the mass of a black-hole, the acceleration of a uniformly accelerating observer, the amplitude of a gravitational wave and the expansion parameter in a cosmological model. In all these examples the fundamental dependance can be seen as the inverse proportionality to the four-volume of the measurement. This dependance can be seen explicitly when one makes the measurement region sufficiently small, in agreement with earlier work on the subject [3].

The methods developed here can easily be applied to many situations involving the measurement of gravity. The uncertainty relation (4) can be used to benchmark the optimality of experimental proposals involving high precision measurements of gravitational phenomena. By evaluating the form of the uncertainty relation (4) for particular scenarios, one can ascertain which parameters have the biggest impact on the measurement results. Hence one can better understand the tradeoff between cost and impact when attempting to optimise experimental design.

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